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# THE MOTION OF A RANDOMLY PERTURBED CHAPLYGIN SLEDGE†

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The motion of a Chaplygin sledge [1] on an inclined plane is considered when sliding and rotational friction are present, together with random "white noise" perturbations produced by, for example, translational vibrations of the base. Stochastic equations of motion are set up and the problem of their statistical analysis is considered.

In the case when the plane is horizontal, and the friction and perturbations are small, the analysis is carried out by an averaging method. All finite kinetic energy distributions of the sledge are found. It is shown that a limiting steady-state mode with a $\gamma$-distribution is established. The motion of a skate (which is a special case of a Chaplygin sledge) on an inclined plane when there is no rotational friction is briefly considered. It is shown that when the sliding friction is small the skate will "on average" slowly slip downwards, i.e. the mathematical expectation of the coordinate changes slightly, whereas the "root mean square" slippage will be significant, i.e. the dispersion of the coordinate varies strongly.

In the case of a Chaplygin sledge moving along an inclined plane with arbitrary coefficients of sliding and rotational friction the analysis is performed using the method of orthogonal expansions. Numerical results are presented.

The deterministic version of the problem (i.e. when there are no random perturbations) has been investigated previously [1–5]. In the stochastic version it has been considered under the assumption [6] that the centre of gravity lies on a line passing through the blade and perpendicular to it, and stochastic equations of motion, linearized about the equilibrium position of the deterministic system, were set up and analysed.

**1.** Consider the motion of a rigid body supported on a smooth inclined plane by a blade and two smooth legs, in a homogeneous gravitational field with acceleration $g$.

We introduce a system of coordinates $O\xi\eta$ fixed to the supporting plane, with the $O\xi\eta$ plane coinciding with the supporting plane and the $O\xi$ axis directed along the line of steepest inclination, and a system of coordinates $Axyz$, rigidly attached to the sledge, the $Ax$ axis directed along the blade and the $Ay$ axis parallel to the supporting plane. Both systems of coordinates are right-handed. (Henceforth we use the notation of [5].) We introduce the following notation: $\alpha$, $\beta$, $\delta$ are the coordinates of the sledge centre of gravity $G$ in the fixed system of coordinates, $m$ is the mass of the sledge, $\rho$ is the radius of inertia of the sledge about the axis $Gz' \parallel Az$, $r=\sqrt{(\alpha^2+\rho^2)}$, and $\theta$ is the angle of inclination of the supporting plane to the horizontal.

One can take as generalized coordinates the coordinates $\xi, \eta$ of the sledge in the fixed system of coordinates $O\xi\eta\zeta$ and the angle $\varphi(\bmod d2\pi)$ of rotation of the sledge about the $Az \parallel O\zeta$ axis. We assume that the sledge cannot slip in a direction perpendicular to its plane. Then the

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non-holonomic constraint equation has the form

$$\dot{\xi}\sin\varphi - \dot{\eta}\cos\varphi = 0$$

We shall assume that the system is acted on by dissipative forces with Rayleigh dissipation function

$$\Phi = (m/2)[h(\dot{\xi}^2 + \dot{\eta}^2) + h_1\dot{\varphi}^2]$$

where $h > 0$ and $h_1 \geqslant 0$ are the coefficients of viscous and rotational friction, together with an external perturbing force $\mathbf{F}(t)$ acting in the $O\xi\eta$ plane, with components $F_1$ and $F_2$ in the $O\xi\eta$ plane.

It is convenient to rewrite the equations of motion using the quasicoordinate $\kappa$

$$\kappa = q - \beta\varphi, \quad \xi^{\cdot} = q^{\cdot}\cos\varphi, \quad \eta^{\cdot} = q^{\cdot}\sin\varphi$$

and the dimensionless quantities

$$t^* = \sqrt{\frac{g}{r}}t, \quad \kappa^* = \frac{\kappa}{r}, \quad p_1^* = \frac{r^2\varphi^{\cdot}}{\sqrt{gr}}, \quad p_2^* = \frac{\kappa^{\cdot}}{\sqrt{gr}}, \quad a = \frac{\alpha}{r}, \quad b = \frac{\beta}{r}$$

$$\xi^* = \frac{\xi}{r}, \quad \eta^* = \frac{\eta}{r}, \quad \xi_G^* = \frac{\xi_G}{r}, \quad \eta_G^* = \frac{\eta_G}{r}, \quad k_{11} = \frac{h\beta^2 + h_1}{r\sqrt{gr}}$$

$$k_{12} = \frac{h\beta}{\sqrt{gr}}, \quad k_{22} = h\sqrt{\frac{r}{g}}, \quad \varepsilon F_i^* = \frac{F_i}{mg} \quad (i = 1,2; \ \varepsilon = \text{const})$$

For simplicity we omit the asterisks and denote differentiation with respect to the "new" time coordinate by a dot. The canonical equations of motion then have the form

$$\begin{aligned}
&\varphi^{\bullet} = p_1\\
&p_1^{\bullet} = -ap_1p_2 - a\sin\theta\sin\varphi - k_{11}p_1 - k_{12}p_2 - (\varepsilon a\sin\varphi)F_1 + (\varepsilon a\cos\varphi)F_2\\
&p_2^{\bullet} = ap_1^2 + \sin\theta\cos\varphi - k_{12}p_1 - k_{22}p_2 + (\varepsilon\cos\varphi)F_1 + (\varepsilon\sin\varphi)F_2
\end{aligned} \tag{1.1}$$

When the function $\mathbf{F}(t)$ is known Eqs (1.1) form a closed system describing the motion of the sledge about the centre $A$. They need to be completed by the equations

$$\begin{aligned}
&\xi^{\cdot} = (bp_1 + p_2)\cos\varphi, \quad \eta^{\cdot} = (bp_1 + p_2)\sin\varphi\\
&\xi_G^{\cdot} = -ap_1\sin\varphi + p_2\cos\varphi, \quad \eta_G^{\cdot} = ap_1\cos\varphi + p_2\sin\varphi
\end{aligned} \tag{1.2}$$

which govern the motion of the blade and centre of gravity $G$ relative to the supporting plane $O\xi\eta$. The equation for $\kappa$, $\kappa = p_2$ separates from Eqs (1.1) and will not be considered below.

**2.** Suppose that the external perturbing force vector $\mathbf{F}$ is a vector $\mathbf{V} = [V_1, V_2]^m$ of independent normal white noise with constant two-by-two intensity matrices $\nu = \nu_0\mathbf{E}$, $\nu_0 = \text{const}$, $\mathbf{E} = \text{diag}(1, 1)$ (broad-band, homogeneous and isotropic in the perturbation space). In this case the equations of motion (1.1) form an Ito system of stochastic differential equations (SDEs) (see, e.g. [7]). These equations govern a time-homogeneous diffusion process on the manifold $S^1 \times R^2$—the phase space of the stochastic non-holonomic Chaplygin system that is being considered.

By a theorem of Khas'minskii [8, p. 119] it follows that when $h\varepsilon_{11}\varepsilon_{21} \neq 0$ the stochastic system (1.1) under consideration contains a limiting and steady-state (in the narrow sense) mode. To prove this it is sufficient to take the sledge kinetic energy function $(p_1^2+p_2^2)/2$ to be the Lyapunov function.

Because Eqs (1.1) are separated from Eqs (1.2), systematic and fluctuating drift [9] of the variables $\xi$, $\eta$, $\xi_G$, $\eta_G$ is possible, i.e. the mathematical expectation and variances of these variables can increase without limit.

Despite the fact that the diffusion of the SDEs under investigation is degenerate, these equations have strong solutions and strongly unique solutions, because all the coefficients in the equations are smooth functions of their variables, and the diffusion matrix (i.e. the matrix $\mathbf{B}\nu\mathbf{B}^m$ where $\mathbf{B}$ is the matrix in front of the white noise vector in the equations of motion) depends only on the sledge orientation and is positive and bounded. Moreover, the process governed by these SDEs is regular (i.e. there is no halting or discontinuity in the process), and the Hormander conditions which guarantee the smoothness of the transitional density in inverse variables are satisfied [10].

The fundamental aim of the paper is a statistical analysis of the motion of a Chaplygin sledge.

Suppose the supporting plane is horizontal ($\theta=0$), and that the friction and perturbations are small, i.e. the quantities $k_{ij}$ $(i,\ j=1,\ 2)$ and $\varepsilon^2$ are of the first order of smallness. To apply the averaging method effectively the variables $\varphi$, $p_1$, $p_2$ are replaced by new variables [5] $w(\mathrm{mod}\,2\pi)$, $I$, $w_1$ using the formulae

$$I=\sqrt{p_1^2+p_2^2}\geqslant 0$$

$$w=\varphi+\frac{1}{a}\arcsin\frac{p_1/p_2}{\sqrt{1+(p_1/p_2)^2}},\quad w_1=\frac{1}{2a}\ln\frac{I+p_2}{I-p_2} \tag{2.1}$$

Formally, in order to reduce the equations of motion to a standard form and use the averaging method, one should introduce a slow variable $w_1-It$ to replace the variable $w_1$. However, we shall not do so because the equation for this variable will not be considered.

In unperturbed motion $I$, $w$, $w_1^{\bullet}=I$ are constant.

The equations of motion in the new variables $I$, $w$, $w_1$ are convenient for the application of the averaging method and for brevity will not be given here.

After the averaging procedure for the diffusion equations in standard form [11–13] one must separately average the drift vector and diffusion matrix. This essentially reduces to averaging the coefficients of the product operator of the Markov process described by the SDEs of motion in new (slow) variables. The averaged system describes with sufficient accuracy the behaviour of the exact system over a time interval of order $1/\varepsilon_*$, $\varepsilon_*=\max\{k_{ij},\ \varepsilon^2(i,\ j=1,\ 2)\}$. More precisely, this means that the finite distributions of the state vector of the original system can, as $\varepsilon_*\to 0$, be approximated in any compact region of the phase space, uniformly over a time interval of length $O(1/\varepsilon_*)$, by finite state vector distributions of the averaged system.

In the present case the system of Ito SDEs has the form (using the previous notation for quantities that are now averaged)

$$I^{\cdot}=-k_{22}I+\nu_0a^2/(2I)+V_1,\quad w^{\cdot}=V_2/I \tag{2.2}$$

It is convenient to replace $I$ by introducing the variable $J=I^2/2$ proportional to the kinetic energy of the Chaplygin sledge. The stochastic equation for the variable $J$

$$J^{\cdot}=-2k_{22}J+\nu_0(1+a^2)/2+\sqrt{2J}V_1 \tag{2.3}$$

can be investigated independently.

Note that the averaged system of stochastic equations (2.2) does not, strictly speaking, allow one to consider the equation for $J$ separately, because the initial values $J(t_0)$, $w(t_0)$, $w_1(t_0)$ are, in general, dependent. However, $J(t)$ constitutes a diffusive process, and distributions of the quantity $J$, as one of the components of the state vector, are the same as in the stochastic system (2.3).

For Eq. (2.3) the analysis problem is completely solvable because the corresponding Pugachev equations [7] for the $n$-dimensional characteristic functions $g_n(\lambda_1, \ldots, \lambda_n; t_1, \ldots, t_n)$ are linear and of the first order, and are integrable by the standard method. We write these equations

$$\frac{\partial g_n}{\partial t_n} = \lambda_n(-2k_{22} + i\nu_0\lambda_n)\frac{\partial g_n}{\partial t_n} + \frac{1}{2}i\nu_0(1+a^2)\lambda g_n, \quad i^2 = -1$$
$$g_n(\lambda_1,\ldots,\lambda_n;t_1,\ldots,t_{n-1},t_{n-1}) = g_{n-1}(\lambda_1,\ldots,\lambda_{n-2},\lambda_{n-1}+\lambda_n;t_1,\ldots,t_{n-1}) \tag{2.4}$$
$$n = 2,3,\ldots; \quad g_1(\lambda, t_0) = g_0(\lambda)$$

Here $g_0(\lambda)$ is the characteristic function of the initial value $J(t_0)$ representing the random quantity, independent of the values of the white noise $\mathbf{V}(t)$ when $t > t_0$. Solutions of Eqs (2.4) have the form

$$g_1(\lambda;t) = g_0(\lambda \exp[-2k_{22}(t-t_0)]\Psi^{-1}(\lambda, t-t_0))\Psi^{-(1+a^2)/2}\ (\lambda, t-t_0)$$
$$\Psi(\lambda,\tau) = 1 - i\nu_0\lambda(1-\exp[-2k_{22}\tau])/(2k_{22}) \tag{2.5}$$
$$g_n(\lambda_1,\ldots,\lambda_n;t_1,\ldots,t_n) = g_{n-1}(\lambda_1,\ldots,\lambda_{n-2},\lambda_{n-1} +$$
$$+\lambda_n \exp[-2k_{22}(t_n - t_{n-1})]\Psi^{-1}(\lambda_n, t_n - t_{n-1}); t_1,\ldots,t_{n-1})\Psi^{-(1+a^2)/2}(\lambda_n, t_n - t_{n-1})$$

From this it follows that over the course of time a steady fluctuation mode is established with the one-dimensional characteristic function

$$g_{st}(\lambda) = (1 - i\nu_0\lambda/(2k_{22}))^{-(1+a^2)/2}$$

which corresponds to the $\gamma$-distribution.

Note that holonomic systems perturbed by white noise often have an energy distribution that is a special case of the $\gamma$-distribution: the exponential or $\chi^2$-distribution with $n$ degrees of freedom. For example, for the stochastic Euler–Poinsot equation [14] the steady-state kinetic energy distribution of the body has a $\chi^2$-distribution with three degrees of freedom.

From Eqs (2.3) (or 2.2)) we obtain a system of linear ordinary differential equations for the moments $\mu_n = MJ^n$ (where $M$ is the mathematical expectation operator)

$$\mu_n^{\cdot} = n[-4k_{22}\mu_n + \nu_0(a^2+2n-1)\mu_{n-1}]/2 \tag{2.6}$$

The solution is easily obtained. Here we shall only write out the expression for $\mu_1(t)$ and for the steady-state moments

$$\mu_1(t) = \mu_1(t_0)\exp(-2k_{22}t) + \nu_0(1+a^2)(1-\exp(-2k_{22}t))/(4k_{22})$$
$$\mu_n^{st} = [\nu_0/(4k_{22})]^n(a^2+2n-1)(a^2+2n-3)\ldots(a^2+1)$$

It is clear from (2.2) that there is no drift in the stochastic equation for the angular variable $w$, i.e. the variable $w$ only undergoes diffusion

$$w(t) = \int_{t_0}^{t} \frac{dW_2(\tau)}{I(\tau)} \quad (I > 0)$$

where $W_2(\tau)$ is the standard one-dimensional Wiener process ($V_2 = W_2$).

In the special case $a = 0$ (where the centre of gravity lies on the line passing through the blade and perpendicular to the blade) the exact equations of motion (1.1) have the form

$$\varphi^{\cdot} = p_1, \quad p_1^{\cdot} = -k_{11}p_1 - k_{12}p_2$$
$$p_2^{\cdot} = -k_{12}p_1 - k_{22}p_2 + V', \quad V' = V_1\cos\varphi + V_2\sin\varphi \tag{2.7}$$

Because the perturbations are homogeneous and isotropic the diffusion matrix of the SDE (2.7) has the form diag(0, 0, $\nu_0$). Hence all the distributions of the state vector $[\varphi,\ p_1,\ p_2]^m$ are the same as for the linear system (2.7), where $V'$ is scalar normal white noise with constant intensity $\nu_0$.

All finite distributions of the state vector $[\varphi,\ p_1,\ p_2]^m$ can be obtained from well-known formulae [7]. We merely remark that in the limit when $t\to\infty$ the distributions are normal in the variables $p_1$ and $p_2$, and uniform (along the circle) in the angular variable $\varphi$.

**3.** In the case of a skate ($\alpha=\beta=0$) on an inclined plane and when there is no rotational friction ($h_1=0$) the stochastic equations of motion (1.1) have the form

$$\varphi^{\cdot} = p_1, \quad p_1^{\cdot} = 0, \quad p_2^{\cdot} = \sin\theta\cos\varphi - k_{22}p_2 + V'$$
$$\xi^{\cdot} = p_2\cos\varphi, \quad \eta^{\cdot} = p_2\sin\varphi \tag{3.1}$$

Henceforth, we shall restrict ourselves, for simplicity, to the case when the initial conditions for $\varphi$ and $p_1$ are deterministic: $\varphi(t_0)=\varphi_0,\ p_1(t_0)=p_{10}$. Then the last three equations of (3.1) are linear, and the evolution of the mathematical expectation $M\xi$ and variance $D\xi$ of the coordinate $\xi$ can be found using well-known formulae [7]. We shall write out expressions for $M\xi$ and $D\xi$ assuming that a periodic regime has been established in the variable $p_2$

$$M\xi = \frac{k_{22}\sin\theta}{p_{10}^2+k_{22}^2}u(t)+c_1, \quad D\xi = \frac{\nu}{p_{10}^2+k_{22}^2}u(t)+c_2$$
$$u(t) = \frac{1}{2}t + \frac{1}{4p_{10}}\sin 2\varphi - \frac{1}{4k_{22}}\cos 2\varphi, \quad \varphi = p_{10}t+\varphi_0$$
$$c_i = \text{const}\ (i=1,2)$$

From this it follows that the ratio of the increments of $D\xi$ and $M\xi$ over one oscillation period $t=2\pi/p_{10}$ of the angle $\varphi$ is equal to

$$(D\xi)_\tau/(M\xi)_\tau = \nu/(k_{22}\sin\theta) \tag{3.2}$$

It is clear that for small $k_{22}$ the skate will "on average" slide downwards very slowly, whereas the "mean square" sliding will be substantial. This result can be considered to be a stochastic analogue of a known effect in the deterministic problem: when there are no perturbations and friction the skate never slides downwards (so long as $\varphi^{\cdot}(t_0)\neq 0$).

**4.** To analyse the motion of a stochastic Chaplygin sledge on an inclined plane with arbitrary coefficients of sliding and rotation friction we shall use an approximate method [15]. Following it, we reduce the problem of finding a one-dimensional density distribution $f(t\colon \varphi,\ p_1,\ p_2)\colon R^1\times S\times R^2\to R$ for solutions of the SDE (1.1) to the problem of finding expansion coefficients for this density in the following series

$$f \sim (2\pi)^{-2}(\gamma_1\gamma_2)^{\frac{1}{2}}\exp[-p_1^2/(2\gamma_1)-p_2^2/(2\gamma_2)]\times$$
$$\times\sum_{i,j=0}^{\infty} H_i(\gamma_1,p_1)H_j(\gamma_2,p_2)\left[d_{ij}+\sqrt{2}\left(\sum_{n=1}^{\infty}a_{ijn}\sin n\varphi+b_{ijn}\cos n\varphi\right)\right]\ (d_{00}=1) \tag{4.1}$$

Here $\gamma_1,\ \gamma_2$ are positive constants chosen for convenience, and $H_i$ are Hermite polynomials.

The unknown coefficients $a_{ijn}(t)$, $b_{ijn}(t)$, $d_{ij}(t)$ are governed by the formulae

$$a_{ijn} = MH_i(\gamma_1, p_1)H_j(\gamma_2, p_2)\sqrt{2}\sin n\varphi$$
$$b_{ijn} = MH_i(\gamma_1, p_1)H_j(\gamma_2, p_2)\sqrt{2}\cos n\varphi$$
$$d_{ij} = MH_i(\gamma_1, p_1)H_j(\gamma_2, p_2)$$

and satisfy the following denumerable infinite system of ordinary differential equations with appropriate initial conditions

$$\begin{aligned}
a_{ijn}^{\cdot} &= -(k_{11}i + k_{22}j)a_{ijn} + n\sqrt{\gamma_1}(\sqrt{i+1}b_{i+1,j,n} + \sqrt{i}b_{i-1,j,n}) - \\
&-a\sqrt{\gamma_2}\left[\sqrt{j}\left(i\left(1-2\frac{\gamma_1}{\gamma_2}\right)-\frac{\gamma_1}{\gamma_2}\right) a_{i,j-1,n} + \sqrt{i(i-1)j}\left(1-\frac{\gamma_1}{\gamma_2}\right)a_{i-2,j-1,n} + i\sqrt{j+1}a_{i,j+1,n} + \right.\\
&\left.+\sqrt{i(i-1)(j+1)}a_{i-2,j+1,n} - \frac{\gamma_1}{\gamma_2}\sqrt{(i+1)(i+2)j}\; a_{i+2,j-1,n}\right] + \\
&+\frac{1}{2}\sin\theta\left[\sqrt{\frac{j}{\gamma_2}}(a_{i,j-1,n+1} + a_{i,j-1,n-1}) - a\sqrt{\frac{i}{\gamma_1}}(b_{i-1,j,n-1} - b_{i-1,j,n+1})\right] - \\
&-k_{12}\left[\sqrt{i\frac{\gamma_2}{\gamma_1}}(\sqrt{j}a_{i-1,j-1,n} + \sqrt{j+1}a_{i-1,j+1,n}) + \sqrt{j\frac{\gamma_1}{\gamma_2}}(\sqrt{i+1}a_{i+1,j-1,n} + \sqrt{i}a_{i-1,j-1,n}) + \right.\\
&\left.+\sqrt{i(i-1)}\left(-k_{11} + \frac{\nu_0 a^2}{2\gamma_1}\right)a_{i-2,j,n} + \sqrt{j(j-1)}\left(-k_{22} + \frac{\nu_0}{2\gamma_2}\right) a_{i,j-2,n}\right. \\
b_{ijn}^{\cdot} &= -(k_{11}i + k_{22}j)b_{ijn} - n\sqrt{\gamma_1}(\sqrt{i+1}a_{i+1,j,n} + \sqrt{i}a_{i-1,j,n}) - \qquad (4.2)\\
&-a\sqrt{\gamma_2}\left\{\sqrt{j}\left[i\left(1-2\frac{\gamma_1}{\gamma_2}\right)-\frac{\gamma_1}{\gamma_2}\right]b_{i,j-1,n} + \sqrt{i(i-1)j}\left(1-\frac{\gamma_1}{\gamma_2}\right) b_{i-2,j-1,n} + i\sqrt{j+1}b_{i,j+1,n} + \right.\\
&\left.+\sqrt{i(i-1)(j+1)}b_{i-2,j+1,n} - \frac{\gamma_1}{\gamma_2}\sqrt{(i+1)(i+2)j}b_{i+2,j-1,n}\right\} + \\
&+\chi\left\{\frac{1}{2}\sin\theta\left[a\sqrt{\frac{i}{\gamma_1}}(-a_{i-1,j,n+1} + a_{i-1,j,n-1}) + \sqrt{\frac{j}{\gamma_2}}(b_{i,j-1,n-1} + b_{i,j-1,n+1})\right]\right\} - \\
&-k_{12}\left[\sqrt{i\frac{\gamma_2}{\gamma_1}}(\sqrt{j}b_{i-1,j-1,n} + \sqrt{j+1}b_{i-1,j+1,n}) + \sqrt{j\frac{\gamma_1}{\gamma_2}}(\sqrt{i+1}b_{i+1,j-1,n} + \sqrt{i}b_{i-1,j-1,n})\right] + \\
&+\sqrt{i(i-1)}\left(-k_{11} + \frac{\nu_0 a^2}{2\gamma_1}\right)b_{i-2,j,n} + \sqrt{j(j-1)}\left(-k_{22} + \frac{\nu_0}{2\gamma_2}\right)b_{i,j-2,n}, \quad \chi = 1
\end{aligned}$$

Equations for $d_{ij}$ are obtained from the equations for $b_{ijn}$ when $n=0$ if one puts $\chi = 1/\sqrt{(2)}$. Equations for the mathematical expectations in $O\xi\eta$ coordinates of the blade $A$ and of the centre of gravity $G$ follow from (1.2)

$$(M\xi)^{\cdot} = b\sqrt{\gamma_1/2}b_{101} + \sqrt{\gamma_2/2}b_{011}, \quad (M\eta)^{\cdot} = b\sqrt{\gamma_1/2}a_{101} + \sqrt{\gamma_2/2}a_{011}$$
$$(M\xi_G)^{\cdot} = -a\sqrt{\gamma_1/2}a_{101} + \sqrt{\gamma_2/2}b_{011}, \quad (M\eta_G)^{\cdot} = a\sqrt{\gamma_1/2}b_{101} + \sqrt{\gamma_2/2}a_{011}$$

Restricting ourselves in (4.2) to $i$, $j$, $n$ such that $0 \leqslant i \leqslant N_1$, $0 \leqslant j \leqslant N_2$, $0 \leqslant n \leqslant N_3$ (the rectangular summation method) and solving the resulting system of linear differential equations on (say) a computer, we obtain an approximate expression for the density distribution, and also approximate values for the moments of the process $[\varphi, p_1, p_2]^T$.

In numerical experiments the values of the following parameters were fixed: $\beta = 0.1$; $r = 1$; $g = 9.8$; $v_0 = 1$; $\sin\theta = 0.1$. The values of the remaining parameters were varied, and we consider three cases: (1) $\alpha = 0.1$; $h = 2.4$; $h_1 = 2$; (2) $\alpha = 0; h = 4; h_1 = 2$; (3) $\alpha = 0; h = 4; h_1 = 0.01$.

When there are no random perturbations the final motion of the sledge in the first case is steady descent along some straight line with constant velocity ($\varphi = \text{const}$, $p_1 = 0$, $p_2 = \text{const}$), in the second case a position of stable equilibrium ($\varphi = 3\pi/2$, $p_1 = p_2 = 0$), and in the third case a self-oscillating mode (which corresponds to a limit cycle in the phase space $\{\varphi, p_1, p_2\}$).

The initial values of $\varphi$, $p_1$, $p_2$ were chosen to be statistically independent. The distribution of the $p_j$ $(j = 1, 2)$ at $t = 0$ was normal with zero mathematical expectation and variance $\gamma_j$ $(j = 1, 2)$, while the distribution of the angle $\varphi$ at $t = 0$ in each of the three cases was as follows:

1. $f_0(\varphi) = [1 + \cos(\varphi - \pi/2)]/(2\pi)$, 2. $f_0(\varphi) = [1 + \cos(\varphi - 5\pi/4)]/(2\pi)$,
3. $f_0(\varphi) = 1/(2\pi)$.

System (4.2) with $N_1 = N_2 = 32$, $N_3 = 6$ was integrated by a fourth-order Runge–Kutta method with step length $10^{-3}$ over the time interval $[0, 10^3]$.

The parameters $\gamma_j$ were chosen to be approximately equal to the steady-state values $Mp_j^2$ and had the following numerical values in the cases under consideration: (1) $\gamma_1 = 0.05$; $\gamma_2 = 0.7$; (2) $\gamma_1 = 0.01$; $\gamma_2 = 0.4$; and (3) $\gamma_1 = 1.56$; $\gamma_2 = 0.41$. Here all the real parts of the eigenvalues of the matrix of the linear system (4.2) were negative, and as $N_1$, $N_2$, $N_3$ increased the Parseval series

$$\sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \sum_{n=1}^{N_3} (d_{ij}^2 + a_{ijn}^2 + b_{ijn}^2)$$

converged fairly rapidly over the time interval [0, 1000] and when $N_1 N_2 \geqslant 32$, $N_3 \geqslant 6$ it remained constant to within $10^{-2}$ This ensured the accuracy of the method.

The numerical results are shown in Figs 1–4. The first set of parameter values corresponds to curve 1 in Fig. 1, the second to curve 2 and the third to Figs 2–4.

Figure 1 shows curves of $[M\xi(t), M\eta(t)]$ when $t \in [0, 1000]$ and the initial values are zero. They illustrate the "average" track of the skate $A$ on the supporting plane. Here the evolution of the one-dimensional density of the angle $\varphi$

$$f(\varphi) = \frac{1}{2\pi}\left[1 + \sum_{n=1}^{N_3} (a_{00n} \sin n\varphi + b_{00n} \cos n\varphi)\right]$$

is also shown. The dashed curves correspond to the initial distribution, and the solid curves to the steady-state distribution. It is clear that the maximum probability shifts from $\pi/2$ to $\varphi^* = 5.65$ in the first case, and from $5\pi/4$ to $3\pi/2$ in the second.

It follows from the form of curves 2 in Fig. 1 that in the second case the sledge will on average slide slowly downwards along the line of steepest descent (systematic drift). Here the skate will oscillate randomly near the $\varphi = 3\pi/2$ position (a stable equilibrium position in the deterministic problem).

Figure 2 shows the projection of the limit cycle on the $p_1$, $p_2$ plane when there are no perturbations, and Fig. 3 shows the steady-state one-dimensional density $f(p_1, p_2)$. It is remarkable that the steady-state one-dimensional density for the variable $p_2$ is essentially non-Gaussian, which is shown in Fig. 4 where the dashed curve corresponds to the normal distribution with the same variance as $p_2$, and the solid curve corresponds to $f(p_2)$. At the same time the steady-state one-dimensional distribution density $f(p_1)$ is practically indistinguishable from the corresponding Gaussian distribution.

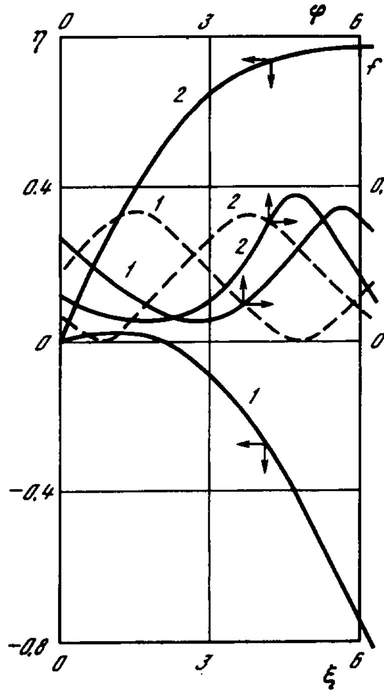


Fig. 1.

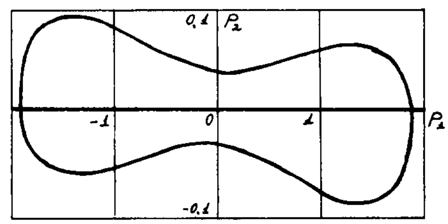


Fig. 2.

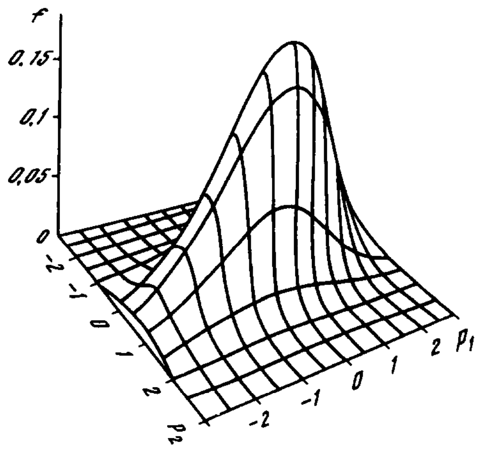


Fig. 3.

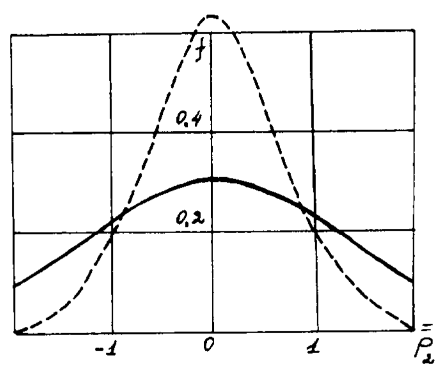


Fig. 4.

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